Zero-Freeness For Mean-Field Spin Glasses

Recall: We consider mixed p-spin models, with Hamiltonian

$$\mathcal{H}_G(\sigma) := \sum_{p=2}^{p_{max}} \frac{\gamma_p}{n^{\frac{p-1}{2}}} \sum_{i_1,\dots,i_p=1}^{n} G_{i_1,\dots,i_p} \prod_{j=1}^{p} \sigma_{i_j},$$

where $G_{i_1,\dots,i_p} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and $\sigma \in \mathbb{R}^n$. For inverse temp. $\beta > 0$, we take our Gibbs measure to be:

$$d\mu_{G,\beta}(\sigma) := \exp(\beta \mathcal{H}_G(\sigma))\, d\rho(\sigma).$$

for ρ uniform on either:

1) $\mathcal{C}_n = \{\pm 1\}^n$ (Ising case),
2) $\mathcal{S}_n = \{\sigma \in \mathbb{R}^n : \|\sigma\|_2^2 = n\}$ (spherical case).

Moreover, we define a mixture function

$$\xi(s) := \sum_{p=2}^{p_{max}} \gamma_p^2 s^p.$$

$$\mathbb{E}_G[\mathcal{H}_G(\tau)\mathcal{H}_G(\sigma)] = n \cdot \xi\left(\frac{\langle \tau, \sigma \rangle}{n}\right)$$

We are interested in the zeros of

$$Z_G(\beta) := \mathbb{E}_{\sigma \sim \rho}[\exp(\beta \mathcal{H}_G(\sigma))].$$

Indeed, last week, Milos showed that if $Z_G(\beta)$ is zero-free on a suff. large disk about the origin, then we can approximate its free energy $\log Z_G(\beta)$ via a Taylor series with $O(\log n/\eta)$ many terms and $\pm\eta$ additive error. We first defined

$$h(m) := \begin{cases} -\frac{1}{2}\left((1-m)\log(1-m) + (1+m)\log(1+m)\right), & \rho = \mathrm{Unif}(\mathcal{C}_n), \\ \frac{1}{2}\log(1-m^2), & \rho = \mathrm{Unif}(\mathcal{S}_n). \end{cases}$$

If $\beta \geq 0$ and $\Psi(m) := \beta^2 \xi(m) + h(m)$, then

$$\beta_{2nd}(e, \xi) := \sup\{\beta \geq 0 : \Psi''(0) < 0 \text{ and } \Psi(m) < 0,\ m \in [-1,1] \setminus \{0\}\}.$$

For this threshold, [BHLLR '25] shows the following.

<u>thm:</u> For any $\varepsilon \in (0,1)$, $\exists\ \delta = \delta(n, \varepsilon) \to 0^+$ as $n \to \infty$ such that w.p. $1-\delta$ over G, $Z_G(\beta)$ is nonzero on the disk $\mathbb{D}(0, (1-\varepsilon)\beta_{2nd})$.

Crucially, such zero-freeness gives access to a poly. time algorithm for approximating $Z_G(\beta)$ up to arbitrary accuracy, as shown by Viles.

However, the result does <u>not</u> follow simply applying Jensen's formula to $Z_G(\beta)$. By Jensen, Viles showed that for any $f_G : \Omega \to \mathbb{C}$ G-measurable that is analytic w/ $f_G(0) \neq 0$, then for $R = (1-\frac{\varepsilon}{2})\beta_{2nd}$, $r = (1-\varepsilon)\beta_{2nd}$,

$$\mathbb{E}_G[\#\{w \in \mathbb{D}(0,r) : f_G(w) = 0\}] \leq$$

Need this to be $1 + o_n(1)$!

$$\frac{1}{2\varepsilon} \mathbb{E}_\theta \log \mathbb{E}_G \left| \frac{f_G(Re^{i\theta})}{f_G(0)} \right|^2 .$$

If we simply take $f_G(\beta) = Z_G(\beta)$, then we would get, with some algebra:

$$\mathbb{E}_G[\#\{w : \mathbb{D}(0,r) : f_G(w) = 0\}] \leq$$

$\xi''(0) = 2\gamma_2^2$

$$\frac{1}{2\varepsilon} \log\left((1 - R^2 \xi''(0))^{-1/2} + o_n(1)\right).$$

This is only useful (at least to conclude zero-freeness via Markov) if

$\gamma_2 = 0$, i.e., no quadratic interactions! This should hint at our desired reweighting, which should satisfy:

$$X_G(\beta) := A_G(\beta) Z_G(\beta) = 1 + o_n(1)$$

$A_G(\beta)$ should be analytic with $A_G(0) = 1$. If so, we would obtain

$$\mathbb{E}_G [\# \{ w \in \mathbb{D}(0,r) : X_G(w) = 0 \}] \leq o_n(1),$$

and hence,

$$\mathbb{P}[\# \{ w \in \mathbb{D}(0,r) : Z_G(w) = 0 \} \geq 1] \leq \mathbb{P}[\# \{ w : \mathbb{D}(0,r) : X_G(w) = 0 \} \geq 1]$$
$$\leq o_n(1),$$

i.e., $Z_G(w)$ is zero-free on $\mathbb{D}(0,r) = \mathbb{D}(0, (1-\varepsilon)\beta_{2nd})$ w.h.p.!

Cluster Expansion

To find an appropriate $A_G(\beta)$, we carry out a cluster expansion in the quadratic part of G. Specifically, we provide a heuristic derivation in Ising case; we anticipate the spherical case to be the same in the second moment phase because the overlap distributions are similar. Now,

$$Z_G(\beta) = \frac{1}{2^n} \sum_{\sigma \in \Sigma^n} \exp(\beta \mathcal{H}_G(\sigma)).$$

We then need $A_G(\beta) \approx Z_G(\beta)^{-1}$. We begin by writing:

$$\mathcal{H}_G(\sigma) := \frac{1}{2} \langle \nabla^2 \mathcal{H}_G(0)\sigma, \sigma \rangle + \mathcal{H}_{G \geq 3}$$

$$= \frac{1}{2} \langle M\sigma, \sigma \rangle + \mathcal{H}_{G \geq 3}. \quad (1)$$

We can explicitly compute

$$M := \nabla^2 \mathcal{H}_G(0) = \frac{\gamma_2}{\sqrt{n}} (G_{i,j} + G_{j,i})_{i,j=1}^n$$

$$= \frac{\xi''(0)^{1/2}}{\sqrt{2n}} \left(G_{i,j} + G_{j,i} \right)_{i,j=1}^{n} .$$

Note that entries of M satisfy:

$$M_{i,j} \sim \begin{cases} \mathcal{N}(0, \frac{\xi''(0)}{n}), & i \neq j, \\ \mathcal{N}(0, 2\frac{\xi''(0)}{n}), & i = j. \end{cases}$$

We also take $z := z(\beta) = \beta \, \xi''(0)^{1/2}$, which satisfies:

<u>Fact:</u> $z(\beta_{2nd}) = \beta_{2nd} \, \xi''(0) \leq 1$. In the defn of β_{2nd}, $h''(0) = -1$, so $0 \geq \varphi''(0) = \beta_{2nd}^2 \, \xi''(0) - 1$.

Finally, note that $\mathcal{H}_{G\geq 3}$ is a spin glass with mixture function

$$\xi_{\geq 3}(s) = \sum_{p=3}^{p_{max}} \gamma_p^2 s^p = \xi(s) - \frac{1}{2} \xi''(0) s^2 .$$

We begin by simply integrating (1) over all terms degree 3 and higher, anticipating that all such terms will only contribute $1 + o_n(1)$ multiplicative fluctuations upon exponentiating. Specifically, observe that $\mathcal{H}_{G\geq 3}(\sigma)$ is a central Gaussian process w/ covariance $n \, \xi_{\geq 3}\left(\frac{\langle \sigma, \tau \rangle}{n}\right)$.

Using the usual formula for the MGF of a Gaussian,

Apply Hoeffding's inequality to see $\frac{\langle \sigma, \tau \rangle}{n}$ concentrates about $\frac{1}{\sqrt{n}}$.

$$\mathbb{E}_{G\geq 3}\left[\exp\left(\beta \mathcal{H}_{G\geq 3}(\sigma) \right) \right] = \exp\left(\frac{n\beta^2}{2} \xi_{\geq 3}(1) \right).$$

Hence, we approximate

$$Z_G(\beta) = \frac{1}{2^n} \sum_{\sigma \in \mathcal{C}_n} \exp\left(\frac{\beta}{2} \langle M\sigma, \sigma \rangle \right) \exp\left(\beta \mathcal{H}_{G\geq 3}(\sigma) \right)$$

$$\approx \frac{1}{2^n} \sum_{\sigma \in \mathcal{C}^n} \exp\left(\frac{\beta}{2} \langle M\sigma, \sigma \rangle \right) \exp\left(\frac{n\beta^2}{2} \xi_{\geq 3}(1) \right)$$

$$= \exp\left(\frac{n\beta^2}{2} \xi(1) - \frac{n z^2}{4} \right) \frac{1}{2^n} \sum_{\sigma \in \mathcal{C}^n} \exp\left(\frac{\beta}{2} \langle M\sigma, \sigma \rangle \right).$$

We can now expand the remaining sum containing degree two interactions via a cluster expansion. To describe the resulting interaction term, we define, for some $K = K(n) > 0$:

edges of K_n

1) $UC(n;K) := \left\{ \Gamma \subseteq \binom{[n]}{2} : \Gamma = \text{edges of vertex disjoint union of cycles with } |\Gamma| \leq K \right\}$,

2) $Ev(n;K) := \left\{ \Gamma \subseteq \binom{[n]}{2} : \Gamma = \text{edges of graph with even degrees}, |\Gamma| \leq K \right\}$,

3) $Cy(n;K) := \left\{ \gamma \subseteq \binom{[n]}{2} : \gamma = \text{edges of a cycle}, |\gamma| \leq K \right\}$.

Now, we can write

$$\exp\left(\frac{\beta}{2} \langle M\sigma, \sigma \rangle \right) = \exp\left(\frac{\beta}{2} \sum_{i=1}^{n} M_{i,i} \right) \cdot \exp\left(\beta \sum_{1 \leq i < j \leq n} M_{i,j} \sigma_i \sigma_j \right)$$

$$= \exp\left(\frac{\beta}{2} \operatorname{Tr}(M) \right) \cdot \prod_{1 \leq i < j \leq n} \exp\left(\beta M_{i,j} \sigma_i \sigma_j \right).$$

Because for $s \in \{\pm 1\}$, $e^{sx} = \cosh(x) + s\sinh(x) = \cosh(x)(1 + s\tanh(x))$,

$$\frac{1}{2^n} \sum_{\sigma \in \mathcal{C}_n} \exp\left(\frac{\beta}{2} \langle M\sigma, \sigma \rangle \right)$$

$$= \exp\left(\frac{\beta}{2} \operatorname{Tr}(M) \right) \cdot \prod_{1 \leq i < j \leq n} \cosh(\beta M_{i,j}) \cdot$$

$$\frac{1}{2^n}\sum_{\sigma\in\mathcal{C}_n}\prod_{1\le i<j\le n}\left(1+\sigma_i\sigma_j\tanh(\beta M_{i,j})\right).$$

$$= \underbrace{\exp\left(\frac{\beta}{2}\operatorname{Tr}(M)\right)}_{(\mathrm{I})}\overbrace{\left(\prod_{1\le i<j\le n}\cosh(\beta M_{i,j})\right)}^{(\mathrm{II})} \cdot \underbrace{\left(\sum_{\Gamma\in \mathrm{Ev}(n)}\prod_{e\in\Gamma}\tanh(\beta M_e)\right)}_{(\mathrm{III})}. \qquad (2)$$

Indeed,

$$\frac{1}{2^n}\sum_{\sigma\in\mathcal{C}_n}\prod_{1\le i<j\le n}\left(1+\sigma_i\sigma_j\tanh(\beta M_{i,j})\right)=$$

$$\frac{1}{2^n}\sum_{\sigma\in\mathcal{C}_n}\sum_{\Gamma\subseteq\binom{[n]}{2}}\left(\prod_{e\in\Gamma}\tanh(\beta M_e)\right)\left(\prod_{i=1}^{n}\sigma_i^{\overset{\text{degree of vertex } i \text{ in } \Gamma}{d_i}}\right).$$

If any degree d_i is odd, then upon summing over $\sigma\in\mathcal{C}_n$, the corresponding term will disappear. If instead d_i is even for all $i\in[n]$, then

$$\sum_{\sigma\in\mathcal{C}_n}\prod_{i=1}^{n}\sigma_i^{d_i} = 2^n,$$

as desired.

Now, we simply aim to invert the terms in (2) analytically, dropping all resulting terms that contribute $1+o_n(1)$ multiplicative fluctuations.

(II). Note that

$$\exp\left(\frac{1}{2}x^2-\frac{1}{12}x^4+O(x^6)\right)=1+\frac{1}{2}x^2-\frac{1}{12}x^4+O(x^6)+\frac{1}{2}\left(\frac{1}{2}x^2-\frac{1}{12}x^4+O(x^6)\right)^2+\dots$$

$$=1+\frac{1}{2}x^2+\frac{1}{24}x^4+O(x^6)$$

$$\approx \cosh(x).$$

Thus, we anticipate

$$\left(\prod_{1 \le i < j \le n} \cosh(\beta M_{i,j}) \right)^{-1}$$

$$= \exp\left(-\frac{1}{2} \sum_{1 \le i < j \le n} (\beta M_{i,j})^2 + \frac{1}{12} \sum_{1 \le i < j \le n} (\beta M_{i,j})^4 + \sum_{1 \le i < j \le n} O(M_{i,j}{}^6) \right)$$

$$\approx \exp\left(-\frac{\beta^2}{2} \sum_{1 \le i < j \le n} M_{i,j}{}^2 + \frac{\beta^4}{12} \sum_{1 \le i < j \le n} M_{i,j}{}^4 \right).$$

Indeed, $M_{i,j} \sim \mathcal{N}(0, \frac{\xi''(0)}{n})$ so the following fact applies:

$$\mathbb{E}[M_{i,j}{}^{2k}] = (2k-1)!! \left(\frac{\xi''(0)}{n} \right)^k = O(n^{-k}),$$

$$\mathrm{Var}(M_{i,j}{}^{2k}) = \mathbb{E}[M_{i,j}{}^{4k}] - \mathbb{E}[M_{i,j}{}^{2k}]^2$$

$$= O(n^{-2k}).$$

Without worrying too much about constants, we see that

$$S_{n,2k} := \sum_{1 \le i < j \le n} (\beta M_e)^{2k}$$

is such that

$$\mathbb{E}[S_{n,2k}] = \binom{n}{2} \beta^{2k} \mathbb{E}[M_e{}^{2k}]$$

$$\asymp n^2 n^{-k}$$

$$= \begin{cases} O(1), & k = 2, \\ O(\frac{1}{n}), & k \ge 3. \end{cases}$$

Similarly, by independence of edges, $\mathrm{Var}(S_{n,2k}) \asymp n^2 n^{-2k} = O(1/n^2)$, so
$$\mathbb{P}\left(|S_{n,4} - \mathbb{E}[S_{n,4}]| \geq \frac{1}{\sqrt{n}} \right) \lesssim \frac{1}{n},$$
for instance.

It follows that

$$\frac{\beta^4}{12} \sum_{1 \le i < j \le n} M_{i,j}^4 = \frac{\beta^4}{12} \sum_{1 \le i < j \le n} \frac{3\xi''(0)^2}{n^2} + o_n(1)$$

$\mathbb{E}[M_{i,j}^4] = 3\left(\frac{\xi''(0)}{n}\right)^2$

$$= \frac{z^4}{8} + o_n(1),$$

and, w.h.p. up to $1 + o_n(1)$ mult. fluctuations,

$$\left(\prod_{1 \le i < j \le n} \cosh(\beta M_{i,j}) \right)^{-1} \approx \exp\left(-\frac{\beta^2}{2} \sum_{1 \le i < j \le n} M_{i,j}^2 \right) \exp\left(\frac{z^4}{8} \right).$$

(III) For the last term, observe that for $e \in E_V(n)$,

$$\tanh(\beta M_e) = \beta M_e - \frac{\beta^3}{3} M_e^3 + \dots$$

$$= \beta M_e + o_n(1),$$

so we aim to make the approximation

$$\sum_{\Gamma \in E_V(n)} \prod_{e \in \Gamma} \tanh(\beta M_e) \approx \sum_{\Gamma \in E_V(n;K)} \prod_{e \in \Gamma} (\beta M_e).$$

Rmk: The above approximation crucially uses the fact from [ALR '87] that, in the high temp. regime, large clusters have exponentially decaying weight. We can observe this phenomenon here as well.

Each factor $\tanh(\beta M_e)$ is mean zero, the above sum is mean zero, and for $|\Gamma| \geq K$, following [ALR '87, Lemma 3.3]:

$$\mathbb{E}\left[\left(\sum_{\substack{\Gamma \in Er(n), \\ |\Gamma| \geq K}} \prod_{e \in \Gamma} \tanh(\beta M_e)\right)^2\right]$$

[independent edges] $$= \sum_{\substack{\Gamma \in Er(n), \\ |\Gamma| \geq K}} \prod_{e \in \Gamma} \mathbb{E}[\tanh(\beta M_e)^2]$$

[$|\tanh(x)| \leq |x|$] $$\leq \sum_{\substack{\Gamma \in Er(n), \\ |\Gamma| \geq K}} \prod_{e \in \Gamma} \mathbb{E}[(\beta M_e)^2]$$

$$= \sum_{\substack{\Gamma \in Er(n), \\ |\Gamma| \geq K}} \prod_{e \in \Gamma} \left(\frac{z^2}{n}\right)$$

[Take $\varepsilon(k)$ s.t. $e^{\varepsilon} z^2 = 1 - \frac{1}{\sqrt{2k}}$] $$\leq e^{-\varepsilon K} \prod_{\gamma \in Cy(n)} \left(1 + \left(\frac{z^2}{n}\right)^{|\gamma|} e^{\varepsilon |\gamma|}\right)$$

[$1 + x \leq e^x$] $$\leq e^{-\varepsilon K} \exp\left(\sum_{\gamma \in Cy(n)} \left(\frac{z^2 e^{\varepsilon}}{n}\right)^{|\gamma|}\right)$$

[Partition over $|\gamma| = m$] $$= e^{-\varepsilon K} \exp\left(\sum_{m=3}^{n} \frac{n(n-1)\ldots(n-m+1)}{2m} \left(\frac{z^2 e^{\varepsilon}}{n}\right)^m\right)$$

$$\leq e^{-\varepsilon K} \exp\left(\sum_{m=3}^{\infty} \frac{(z^2 e^{\varepsilon})^m}{2m}\right)$$

$$\leq z^{2K} (e^{\varepsilon} z^2)^{-K} \exp\left(\frac{1}{2} \sum_{m=0}^{\infty} (z^2 e^{\varepsilon})^m\right)$$

$$= z^{2K} \exp\left(-k \log(e^{\varepsilon} z^2) + \frac{1}{2} \cdot \frac{1}{1 - e^{\varepsilon} z^2}\right)$$

$\frac{\sqrt{2k}}{2}$

[Use def. $e^{\varepsilon} z^2 = 1 - \frac{1}{\sqrt{2k}}$] $$\leq C z^{2K} \exp(\sqrt{2K}).$$ $-\log(1-y) \leq y + \frac{y^2}{2(1-y)}$, $y \in (0,1)$

As $z = \beta \xi''(0)^{1/2} < 1$ for $\beta < \beta_{2nd}$, if $K = K(n) \to \infty$ as $n \to \infty$, then this is $o_n(1)$! We will take $K = \log \log n$; with this choice, we can safely discard higher order terms of $\tanh(\beta M_e)$.

Now, consider any graph $\Gamma \subseteq \binom{[n]}{2}$. We may take

$$C_\Gamma(\beta) = \sum_{\Gamma' \sim \Gamma} \prod_{e \in \Gamma'} (\beta M_e),$$

which has $\mathbb{E}[C_\Gamma(\beta)] = 0$ and

$$\begin{aligned}
\mathbb{E}[C_\Gamma(\beta)^2] &= \mathbb{E} \sum_{\Gamma_1 \sim \Gamma} \sum_{\Gamma_2 \sim \Gamma} \left(\prod_{e_1 \in \Gamma_1} \beta M_{e_1} \right) \left(\prod_{e_2 \in \Gamma_2} \beta M_{e_2} \right) \\
&= \sum_{\Gamma' \sim \Gamma} \prod_{e \in \Gamma'} \mathbb{E}[\beta^2 M_e^2] \\
&= \sum_{\Gamma' \sim \Gamma} O\left(n^{-|E(\Gamma)|}\right) \\
&= \frac{n(n-1)\dots(n-|V(\Gamma)|+1)}{|\mathrm{Aut}(\Gamma)|} O\left(n^{-|E(\Gamma)|}\right) \\
&= O\left(n^{|V(\Gamma)|-|E(\Gamma)|}\right).
\end{aligned}$$

$\left[\mathbb{E}[\beta^2 M_e^2] = \frac{\beta^2 \xi''(0)}{n} = \frac{z}{n},\right.$ and $z<1$ for $\left.\beta < \beta_{2nd}\right]$

Hence, if $\Gamma \in \mathcal{UC}(n)$, total contribution is $O(1)$. If $\Gamma \in \mathcal{E}v(n) \setminus \mathcal{UC}(n)$, then at least one vertex is repeated, and $C_\Gamma(\beta)$ contributes $O(1/n)$. With $K = \log\log n$, this decay is sufficient!

$$\sum_{\Gamma \in \mathcal{E}v(n)} \prod_{e \in \Gamma} \tanh(\beta M_e) \approx \sum_{\Gamma \in \mathcal{UC}(n;K)} \prod_{e \in \Gamma} (\beta M_e) \qquad (*)$$

$$\approx \prod_{\Gamma \in \mathcal{C}y(n;K)} \left(1 + \prod_{e \in \Gamma} (\beta M_e)\right),$$

using the same reasoning $(*)$ to see that multigraphs contribute subleading order in the last approximation (see appendix for careful handling of all of this).

Identically,

number of conn. components

$$\sum_{\Gamma \in \mathcal{UC}(n)} (-1)^{C(\Gamma)} \prod_{e \in \Gamma} (\beta M_e) \approx \prod_{\Gamma \in \mathcal{C}y(n;K)} \left(1 - \prod_{e \in \Gamma} (\beta M_e)\right).$$

In turn,

$$\left(\sum_{\Gamma \in Ev(n)} \prod_{e \in \Gamma} \tanh(\beta M_e) \right)^{-1} \approx \prod_{\gamma \in Cy(n;k)} \left(1 + \prod_{e \in \gamma} (\beta M_e) \right)^{-1}$$

$$= \prod_{\gamma \in Cy(n;k)} \left(1 - \prod_{e \in \gamma} (\beta M_e) \right) \cdot$$

$$\prod_{\gamma \in Cy(n;k)} \left(1 - \prod_{e \in \gamma} (\beta M_e)^2 \right)^{-1}$$

$$\approx \left(\sum_{\Gamma \in \mathcal{U}C(n;k)} (-1)^{c(\Gamma)} \prod_{e \in \Gamma} (\beta M_e) \right) \cdot$$

$$\prod_{\gamma \in Cy(n;k)} \left(1 - \prod_{e \in \gamma} (\beta M_e)^2 \right)^{-1}.$$

Finally, we simply estimate: $\nearrow$ $\frac{1}{1-x^2} = \sum_{i=0}^{\infty} (x^2)^i = 1 + x^2 + O(x^4)$
$= e(x^2 + O(x^4))$

$$\prod_{\gamma \in Cy(n;k)} \left(1 - \prod_{e \in \gamma} (\beta M_e)^2 \right)^{-1} \approx \exp\left(\sum_{\gamma \in Cy(n;k)} \prod_{e \in \gamma} (\beta M_e)^2 + \sum_{\gamma \in Cy(n;k)} O\left(\prod_{e \in \gamma} M_e^4 \right) \right)$$

$$\approx \exp\left(\sum_{\gamma \in Cy(n;k)} \left(\frac{z^2}{n} \right)^{|\gamma|} \right)$$

$$= \exp\left(\sum_{k=3}^{n} \frac{n(n-1)\dots(n-k+1)}{2k} \frac{z^{2k}}{n^k} \right)$$

$$\approx \exp\left(\sum_{k=3}^{\infty} \frac{z^{2k}}{2k} \right).$$

Now, $$\sum_{k=3}^{\infty} \frac{z^{2k}}{2k} = \sum_{k=1}^{\infty} \frac{z^{2k}}{2k} - \left(\frac{z^2}{2} + \frac{z^2}{4} \right)$$

$$= -\frac{1}{2} \log(1 - z^2) - \left(\frac{z^2}{2} + \frac{z^2}{4} \right),$$

yielding

$$(\mathrm{III})^{-1} = \Big(\sum_{\Gamma \in E_v(n)} \prod_{e \in \Gamma} \tanh(\beta M_e) \Big)^{-1}$$

$$\approx (1-z^2)^{-1/2} \exp\Big(-\frac{z^2}{2} - \frac{z^4}{4}\Big) \cdot \Big(\sum_{\Gamma \in \mathcal{UC}(n;k)} (-1)^{c(\Gamma)} \prod_{e \in \Gamma} (\beta M_e) \Big)$$

In summary, the final reweighting factor is then:

$$A_G(\beta) := (\mathrm{I})^{-1} (\mathrm{II})^{-1} (\mathrm{III})^{-1} + o_n(1)$$

$$\approx (1-z^2)^{-1/2} \exp\Big(-\frac{n\beta^2}{2} + \frac{nz^2}{4} - \frac{z^2}{2} - \frac{z^4}{4} \Big) \cdot$$

$$\exp\Big(-\frac{\beta}{2} \mathrm{Tr}(M) - \frac{\beta^2}{2} \sum_{i<j} M_{i,j}^2 \Big) \cdot$$

$$\Big(\sum_{\Gamma \in \mathcal{UC}(n;K)} (-1)^{c(\Gamma)} \prod_{e \in \Gamma} (\beta M_e) \Big).$$

As desired, $A_G(\beta)$ is analytic in β on $\overline{\mathbb{D}(0,R)}$, as $z = \beta \xi''(0)^{1/2}$ satisfies $|z| < 1$ on this region. Moreover, $A_G(\beta)$ satisfies $A_G(0) = 1$ so that for $X_G(\beta) := A_G(\beta) Z_G(\beta)$, we can faithfully apply Jensen's formula and use the upper bound

$$\mathbb{E}_G \Big| \frac{X_G(\beta)}{X_G(0)} \Big|^2 \le 1 + o_n(1), \qquad \beta \in \overline{\mathbb{D}(0,R)}$$

to conclude zero-freeness.

Future Directions

1) Genuine polynomial time algorithms up to β_{2nd}?

$\hookrightarrow$ So far no extension of method of Patel and Regts, which works in for various edge-coloring problems, to the mean-field spin glass setting.

2) Can $Z_G(\beta)$ be approximated up to β_{RS}?

3) Even for Sherrington-Kirkpatrick, it is not clear how to handle the presence of an external field:

$$Z_{G,\beta,\lambda}(\sigma) := \mathbb{E}_{\sigma\sim e}[\exp(\beta \mathcal{H}_G(\sigma) + \lambda \cdot \langle \sigma, 1\rangle)]$$

Even beyond this, what if the external field is random? Is this easier or harder to handle?

$\hookrightarrow$ See "A Constructive Proof of the Spherical Parisi Formula" by Huang and Sellke, which describes the limiting free energy per site (including in the low-temp. phase) for mixed p-spin Hamiltonians with a random external field.

Backup: Outline of Rigorous Proof

Prop: Let $X_G(\beta) = Z_G(\beta) A_G(\beta)$, with reweighting $A_G(\beta)$ as above. Then,

$$\mathbb{E}_G\left[\left|\frac{X_G(\beta)}{X_G(0)}\right|^2\right] = 1 + o_n(1), \quad \beta \in \overline{\mathbb{D}(0, R)}.$$

pf:

Step 1) Observe that

$$\mathbb{E}_G\left[\left|\frac{X_G(\beta)}{X_G(0)}\right|^2\right] =$$

$$\mathbb{E}_{\sigma,\tau\sim e}\mathbb{E}_G\left[\exp(\beta \mathcal{H}_G(\sigma) + \overline{\beta}\mathcal{H}_G(\tau))A_G(\beta)A_G(\overline{\beta})\right].$$

Now, note that $f(G) := A_G(\beta)A_G(\overline{\beta})$ becomes, for some poly.

p in G,

$$f(G) = \exp\left(-\mathrm{Re}(\beta)\mathrm{Tr}(M) - \mathrm{Re}(\beta^2)\sum_{i<j} M_{i,j}^2\right) p(G)$$

$$= \exp\left(-\mathrm{Re}(z)\sqrt{\frac{2}{n}}\sum_i G_{i,i} - \frac{\mathrm{Re}(z^2)}{2n}\left(G_{i,j}+G_{j,i}\right)^2\right) p(G).$$

If we take $m = n^2 + n^3 + \dots + n^{p_{max}}$,

$$G = (G_{i_1,\dots,i_p} : 2 \le p \le p_{max},\ 1 \le i_1 < \dots < i_p \le n) \in \mathbb{C}^m,$$

then the following lemma applies, proven via a Gaussian change of measure:

<u>Lemma:</u> Define, for $\beta \in \mathbb{C}$ and $\sigma, \tau \in S_n$, the <u>planted Hamiltonian</u>

$$\mathcal{H}_G^{p,\beta,\sigma,\tau}(x) := \mathcal{H}_G(x) + \beta\xi\left(\frac{\langle x,\sigma\rangle}{n}\right) + \bar{\beta}\xi\left(\frac{\langle x,\tau\rangle}{n}\right).$$

Then, if $f(G) = \exp(\langle AG, G\rangle + \langle v, G\rangle)\, p(G)$, where $A \in \mathbb{R}^{m\times m}$ is symmetric, $A \prec \frac{1}{2} I_m$, $v \in \mathbb{C}^m$, and p is poly.,

$$\mathbb{E}\left[\exp\left(\beta \mathcal{H}_G(\sigma) + \bar{\beta}\mathcal{H}_G(\tau)\right) f(\mathcal{H}_G)\right] =$$

$$\mathbb{E}\left[\exp\left(\beta \mathcal{H}_G(\sigma) + \bar{\beta}\mathcal{H}_G(\tau)\right)\right] \mathbb{E}\left[f(\mathcal{H}_G^{p,\beta,\sigma,\tau})\right].$$

Applying this above, with

$$A_{(i_1,\dots,i_p)(j_1,\dots,j_{p'})} = -\frac{\mathrm{Re}(z^2)}{2n}\mathbb{1}\left[p=p'=2,\ i_1 \ne i_2 \text{ and } \begin{matrix}((i_1,i_2)=(j_1,j_2) \text{ or} \\ (i_1,i_2)=(j_2,j_1)\end{matrix}\right],$$

block 2x2 diagonal

so that $A \preceq \max\left(-\frac{\mathrm{Re}(z^2)}{2n}, 0\right) I \preceq \frac{1}{n} I$, we see that

$$\mathbb{E}\left[\left|\frac{X_G(\beta)}{X_G(0)}\right|^2\right] = \mathbb{E}_{\sigma,\tau\sim\rho}\Big[\mathbb{E}_G\left[\exp\left(\beta \mathcal{H}_G(\sigma) + \bar{\beta}\mathcal{H}_G(\tau)\right)\right] \mathbb{E}^{P,\beta,\sigma,\tau}\left[A_G(\beta)A_G(\bar{\beta})\right]\Big]$$

$$= \underset{\substack{q\sim ov(\rho) \\ \text{overlap dist.}}}{\mathbb{E}}\ \underset{\substack{(\sigma,\tau)\sim\rho_2(q) \\ \text{overlap cond. on} \\ \text{overlap } q}}{\mathbb{E}}\Big[\mathbb{E}_G\left[\exp\left(\beta \mathcal{H}_G(\sigma) + \bar{\beta}\mathcal{H}_G(\tau)\right)\right] \mathbb{E}^{P,\beta,\sigma,\tau}\left[A_G(\beta)A_G(\bar{\beta})\right]\Big]$$

Step 2)

If $(\sigma,\tau)\sim\rho_2(q)$, then

$$\mathbb{E}_G\left[\exp\left(\beta\mathcal{H}_G(\sigma) + \bar{\beta}\mathcal{H}_G(\tau)\right)\right] =$$
$$\exp\left(n\,\mathrm{Re}(\beta^2)\,\xi(1) + n\,|\beta|^2\,\xi(q)\right)$$

On the other hand, if for $W\in\mathbb{C}^{n\times n}$ we define

$$W^{\lambda,\sigma,\tau} = W + \frac{\lambda}{n}\sigma\sigma^T + \frac{\bar{\lambda}}{n}\tau\tau^T,$$

tedious algebra shows that under the planted law,

$$A_G(\beta)A_G(\bar{\beta}) =$$

$$(*)\quad \begin{cases} |1-z^2|^{-1}\exp\Big(-n\,\mathrm{Re}(\beta^2)\,\xi(1) + \frac{n}{2}\mathrm{Re}(z^2) - \mathrm{Re}(z^2) - \frac{1}{4}\mathrm{Re}(z^4)\Big) \cdot \\ \exp\Big(-\mathrm{Re}(z)\,\mathrm{Tr}(W^{z,\sigma,\tau}) - \mathrm{Re}(z^2)\sum_{i<j}\big(W^{z,\sigma,\tau}_{i,j}\big)^2\Big) \\ \cdot\ f_z(W^{z,\sigma,\tau}), \end{cases}$$

where $W\sim GOE(n)$ and

$$f_\lambda(W) = \Big(\sum_{\Gamma\in UC(n)} (-1)^{c(\Gamma)}\prod_{e\in\Gamma}(\lambda W_e)\Big)\cdot \Big(\sum_{\Gamma\in UC(n)} (-1)^{c(\Gamma)}\prod_{e\in\Gamma}(\bar{\lambda} W_e)\Big).$$

If we define $\mathbb{E}_{(\sigma,\tau)\sim e_2(q)} \mathbb{E}_{W\sim GOE(n)} [(*)] =: \mathrm{Val}(z,q)$,

then

$$\mathbb{E}_G\left[\left|\frac{X_G(\beta)}{X_G(0)}\right|^2\right] = \mathbb{E}_{q\sim or(e)}\left[\exp(n|\beta|^2 \xi(q))\, \mathrm{Val}(z,q)\right].$$

Step 3) [BHLLR Prop. 3.4]

For some $\alpha \in (0, 1/4)$, show that

$$\mathbb{E}_{q\sim or(e)}\left[\mathbb{1}[|q|\le n^{-1/2+\alpha}] \exp(n|\beta|^2 \xi(q))\right] \le (1-|z|^2)^{-1/2} + o_n(1),$$

and that

$$\mathbb{E}_{q\sim or(e)}\left[\mathbb{1}[|q| > n^{-1/2+\alpha}] \exp(n|\beta|^2 \xi(q))\right] \le \exp(-\Omega(n^{2\alpha})),$$

Step 4) [BHLLR Prop. 3.5]

Show that for any $\lambda \in \mathbb{D}(0, 1-\frac{\varepsilon}{2})$ and $q \in [-1,1]$,

$$|\mathrm{Val}(\lambda,q)| \le O(2^{4K}) = O(\log n).$$

Moreover, if $q \in \mathrm{supp}\, or(e)$ and $\alpha \in (0, 1/4)$ is as above,

$$\mathrm{Val}(\lambda, q) = (1-|\lambda|^2)^{1/2} + o_n(1).$$

Step 5)

To conclude,

$$\mathbb{E}_G\left[\left|\frac{X_G(\beta)}{X_G(0)}\right|^2\right]$$

$$= \mathbb{E}_{q \sim \mathrm{or}(e)}\left[\exp\left(n|\beta|^2 \xi(q)\right) \mathrm{Val}(z,q)\right].$$

$$= \mathbb{E}_{q\sim \mathrm{or}(e)}\left[\mathbb{1}[|q| \le n^{-1/2+\alpha}] \exp(n|\beta|^2\xi(q)) \mathrm{Val}(z,q)\right]$$

$$+ \mathbb{E}_{q\sim \mathrm{or}(e)}\left[\mathbb{1}[|q| > n^{-1/2+\alpha}] \exp(n|\beta|^2\xi(q)) \mathrm{Val}(z,q)\right]$$

$$[|z| \le 1-\tfrac{\varepsilon}{2}] \le \left((1-|z|^2)^{1/2} + o_n(1)\right) \mathbb{E}_{q\sim\mathrm{or}(e)}\left[\mathbb{1}[|q| \le n^{-1/2+\alpha}] \exp(n|\beta|^2\xi(q))\right]$$

$$+ O(\log n)\, \mathbb{E}_{q\sim\mathrm{or}(e)}\left[\mathbb{1}[|q| > n^{-1/2+\alpha}] \exp(n|\beta|^2\xi(q))\right]$$

$$\le \left((1-|z|^2)^{1/2} + o_n(1)\right)\left((1-|z|^2)^{-1/2} + o_n(1)\right) +$$

$$O(\log n) \exp(-\Omega(n^{2\alpha}))$$

$$= 1 + o_n(1). \qquad \square$$

Backup: High-Order Terms in the Cluster Expansion

In the above derivation, we required that sums of the following form are $o_n(1)$ to safely ignore: for

$$\mathcal{C}(n;K) \subseteq \left\{\Gamma \subseteq \binom{[n]}{2} : |\Gamma| \le K\right\},$$

some class of graphs (e.g., $\mathcal{C}(n;K) = \mathrm{Er}(n;K) \setminus \mathcal{U}\mathcal{C}(n;K)$),

$$\sum_{[\Gamma]\in \mathcal{C}(n;K)/\sim} \; \sum_{\Gamma' \sim \Gamma} \prod_{e\in\Gamma'} (\beta M_e) = \sum_{[\Gamma]\in\mathcal{C}(n;K)/\sim} C_\Gamma(\beta).$$

We showed above that $\mathbb{E}[C_\Gamma(\beta)^2] = O(n^{|V(\Gamma)| - |E(\Gamma)|})$, meaning that its fluctuations are $o_n(1)$, due to Chebyshev, if $\mathcal{C}(n;K)$ is such that $|V(\Gamma)| < |E(\Gamma)|$ for all $\Gamma \in \mathcal{C}(n;K)$.

We can thus investigate the combinatorial sum on the outside.

<u>Lemma:</u> For any set $\mathcal{C}(n;K)$, as defined above, and $K = \log\log n$,

$$|\{[\Gamma] \in \mathcal{C}(n;K)/\sim\}| = o(n).$$

<u>pf:</u> Any $[\Gamma] \in \mathcal{C}(n;K)/\sim$ can be uniquely identified with an isomorphism class of a graph on at most K edges, with no isolated vertices.

Thus, we can bound the number of graphs on up to K edges, up to isomorphism. We first count graphs with exactly m edges, which have at most $2m$ (non-isolated) vertices. Thus, we can get an upper bound if we count graphs on $2m$ vertices (not necessarily non-isolated) and m edges. This yields

$$\binom{\binom{2m}{2}}{m} \le \binom{(2m)^2}{m}$$

$$\le \frac{(2m^2)^m}{m!}.$$

By Stirling's formula, $m! \ge (m/e)^m$, so we obtain

$$\binom{\binom{2m}{2}}{m} \le \frac{(2m^2)^m}{(m/e)^m} = (2em)^m.$$

Summing over all $m \le K$, we get a bound of

$$K(2eK)^K = (2e)^K K^{K+1}.$$

Taking $K = \log\log n$, it follows that

$$|\{[\Gamma] \in \mathcal{C}(n;K)/\sim\}| \leq (2e)^{\log\log n} \cdot (\log\log n)^{\log\log n} \cdot \log\log n$$

$\left[(\log\log n)^{\log\log n} = o(n^{\varepsilon}) \text{ for all } \varepsilon > 0 \right]$

$$\leq O\left((\log n)^2 \cdot n^{0.001} \cdot \log\log n \right)$$

$$= o(n). \qquad \square$$

From this lemma, we can indeed say that

$$\sum_{\Gamma \in Ev(n)} \prod_{e \in \Gamma} \tanh(\beta M_e) = \sum_{\Gamma \in Ev(n;K)} \prod_{e\in\Gamma} (\beta M_e) + o_n(1), \quad \text{w.h.p.},$$

<u>and</u> that

$$\sum_{\Gamma \in Ev(n;K)} \prod_{e\in\Gamma} (\beta M_e) = \sum_{\Gamma \in \mathcal{U}C(n;K)} \prod_{e\in\Gamma} (\beta M_e) + o_n(1), \quad \text{w.h.p.},$$

applying our lemma with (a) $\mathcal{C}(n;K) = Ev(n;K)$,

(b) $\mathcal{C}(n;K) = Ev(n;K) \setminus \mathcal{U}C(n;K)$.

Similarly, terms of the form

$$\sum_{\gamma \in Cy(n;K)} O\left(\prod_{e\in\gamma} M_e^{4} \right) = o_n(1), \quad \text{w.h.p.},$$

using $\mathcal{C}(n;K) = Cy(n;K)$ and that $\mathbb{E}[M_e^4] = O(1/n^2)$.

Now, only one loose end remains in our cluster expansion computations:

$$\sum_{\Gamma \in \mathcal{U}C(n;K)} \prod_{e\in\Gamma} (\beta M_e) = \prod_{\Gamma \in Cy(n;K)} \left(1 + \prod_{e\in\Gamma} (\beta M_e)\right) + o_n(1), \text{ w.h.p.}$$

Indeed, the product on the right yields multigraphs, which require some care to discard. First, we write:

$$\prod_{\Gamma \in Cy(n;K)} \Big(1 + \prod_{e \in \Gamma} (\beta M_e) \Big) = \sum_{\Gamma \in Ev(n) \setminus Ev(n;K)} \prod_{e \in \Gamma} (\beta M_e) +$$

$$\sum_{\Gamma \in Ev(n;K)} \prod_{e \in \Gamma} (\beta M_e) +$$

$$\sum_{\Gamma^* \in Mult(n;K)} \prod_{e \in \Gamma^*} (\beta M_e) +$$

$$\sum_{\Gamma^* \in Mult(n) \setminus Mult(n;K)} \prod_{e \in \Gamma^*} (\beta M_e),$$

where

formed as a product of unique cycles!

$$Mult(n;K) = \{ \Gamma^* \text{ multigraph on } n \text{ vertices with } \leq |K| \text{ total edges} \},$$

and $Mult(n)$ is the obvious constraint-free analogue. Now, the first and last terms are $o_n(1)$ w.h.p. because (1) if $|\Gamma| \geq K$, the contribution to cluster expansion is $o_n(1)$, as seen above, and (2), we can apply the exact same reasoning to see that the last term is also $o_n(1)$. What remains is the third term:

<u>Prop</u>: With $Mult(n;K)$ as defined above and $K = \log \log n$,

$$\sum_{\Gamma^* \in Mult(n;K)} \prod_{e \in \Gamma^*} (\beta M_e) = o_n(1), \quad \text{w.h.p.}$$

<u>pf</u>: As usual, the sum in question has mean zero; any such Γ^* is a product of <u>unique</u> cycles and hence one multi-edge in

Γ^* has multiplicity one. Now,

$$\mathbb{E}\left[\left(\sum_{\Gamma^* \in Mult(n;K)} \prod_{e \in \Gamma^*} (\beta M_e)\right)^2\right] =$$

$$(*) \quad \sum_{\Gamma_1^*, \Gamma_2^*} \mathbb{E}\left[\prod_{e_1, e_2 \in \Gamma_1^*, \Gamma_2^*} \beta^2 M_{e_1} M_{e_2}\right].$$

Again the inner expectation is nonzero iff each edge in $\Gamma_1^* \cup \Gamma_2^*$ is repeated an even number of times. If $\Gamma_1^* \cup \Gamma_2^*$ is of this form, then the total number of edges, with multiplicity, is denoted $|E(\Gamma_1^* \cup \Gamma_2^*)|$, and

$$\mathbb{E}\left[\prod_{e_1, e_2 \in \Gamma_1^*, \Gamma_2^*} \beta^2 M_{e_1} M_{e_2}\right] = O\left(n^{-\frac{|E(\Gamma_1^* \cup \Gamma_2^*)|}{2}}\right),$$

recalling that $\mathbb{E}[M_e^{2k}] = O(n^{-k})$. Now, we partition unions $\Gamma_1^* \cup \Gamma_2^*$ into equivalence classes, with

$$\Gamma_1^* \cup \Gamma_2^* \sim \Gamma_3^* \cup \Gamma_4^*$$

iff the underlying graph topologies (ignoring multiplicities) are the same. Because the underlying graph topologies correspond to graphs on at most $2K$ edges, $(*)$ can be bounded by:

$$\sum_{[\Gamma] \in \mathcal{C}(n;2K)/\sim} \sum_{[\Gamma_1^* \cup \Gamma_2^*] = [\Gamma]} \mathbb{E}\left[\prod_{e_1, e_2 \in \Gamma_1^*, \Gamma_2^*} \beta^2 M_{e_1} M_{e_2}\right]$$

$$\leq \sum_{[\Gamma] \in \mathcal{C}(n;2K)/\sim} \sum_{\Gamma_1^* \cup \Gamma_2^* \sim \Gamma} n^{-\frac{|E(\Gamma_1^* \cup \Gamma_2^*)|}{2}}$$

$$\leq \sum_{[\Gamma] \in \mathcal{C}(n;2k)/\sim} n^{-|E(\Gamma)|} \sum_{\Gamma_1^* \cup \Gamma_2^* \sim \Gamma}$$

$$\leq \sum_{[\Gamma] \in \mathcal{C}(n,2k)/\sim} n^{|V(\Gamma)| - |E(\Gamma)|}$$

Above, $\Gamma_1^*, \Gamma_2^* \in Mult(n;K)$ are such that each edge in $\Gamma_1^* \cup \Gamma_2^*$ has even multiplicity and each vertex has degree at least two (Γ_1^*, Γ_2^* are products of cycles), then

$$\tfrac{1}{2}|E(\Gamma_1^* \cup \Gamma_2^*)| \geq |E(\Gamma)| \geq |V(\Gamma)| + 1,$$

so the summand above is indeed $O(\frac{1}{n})$ for each $[\Gamma] \in \mathcal{C}(n;2k)/\sim$. Finally, as shown above, the combinatorial factor from the outer sum is bounded by

$$(2e)^{2k}(2k)^{2k+1} \leq O((\log n)^4)\, O(2\log\log n) \cdot$$
$$O(2\log\log n^{2\log\log n})$$

$$= \tilde{o}(n^{0.001})$$

so the variance of the terms contributed by $\Gamma^* \in Mult(n;K)$ is indeed $o_n(1)$, and by Chebyshev's inequality, the conclusion follows. $\square$

External Field Calculations

With a Gaussian external field,

$$\mathbb{E}[\beta H_G(\sigma) + \lambda \cdot \langle \sigma, g \rangle] = 0$$

$$\mathbb{E}[(\beta H_G(\sigma) + \lambda \cdot \langle \sigma, g\rangle)(\beta H_G(\tau) + \lambda \cdot \langle \tau, g\rangle)] =$$

$$\beta^2 n \cdot \xi\left(\frac{\langle \tau, \sigma \rangle}{n}\right) + \lambda^2 \mathbb{E}\left[\sum_{i=1}^{n} \sigma_i g_i \cdot \sum_{j=1}^{n} \tau_j g_j\right] =$$

$$n\beta^2 \xi\left(\frac{\langle \tau, \sigma \rangle}{n}\right) + \lambda^2 \sum_{i,j=1}^{n} \sigma_i \tau_j \mathbb{E}[g_i, g_j] =$$

$$n\beta^2 \xi\left(\frac{\langle \tau, \sigma \rangle}{n}\right) + \boxed{\lambda^2 \langle \tau, \sigma \rangle}.$$

The highlighted term has, in general, $O(n)$ fluctuations for $\sigma, \tau \overset{\text{i.i.d.}}{\sim} \rho$. This precludes any usage of the second moment method, at least without serious modifications.